Improved Navigation by Combining VOR/DME Information and Air Data

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Combining VOR/DME information (from one or two stations) and air data (airspeed and heading) by means of a maximum likelihood filter is shown to result in substantial improvements in navigational accuracy. In some cases, rms position errors are only 5% of those obtained when using a single VOR/DME (current practice). Much of this reduction results from estimating the bias errors associated with the VOR/DME system. The use of this filter, implemented by a small airborne computer, results in sufficient accuracy for area navigation, i.e., navigation using flight paths that do not overfly VOR stations.

Introduction

THE primary navigation aid for civil aircraft flying in U.S. airspace is the VOR/DME system. The VOR (Very high-frequency Omni-Range) allows the aircraft to determine its bearing to the station to within about 1.5° (rms error), whereas the DME (Distance Measuring Equipment) allows the aircraft to determine its distance to the station to within about 1000 ft (rms error). This results in an rms azimuthal position error of about three naut miles when 120 naut miles from the station.

Current use of the VOR/DME system involves radial navigation, i.e., aircraft fly directly to or from the ground stations. If simple triangulation computations could be made more or less continously onboard the aircraft, area navigation could be used, i.e., the VOR/DME system could be used without having to fly VOR radials. Area navigation would make it possible to fly more direct flight paths, which would result in decreased flight times, route mileage, fuel requirements, pollution, and operating costs. It would also reduce traffic congestion over the VOR/DME stations and permit closely-spaced parallel tracks to accommodate large numbers of aircraft on busy airways. It also introduces considerable flexibility to fly around bad weather and congested areas.

On the average, position errors are greater for area navigation than for radial navigation. This comes about because the position error resulting from VOR error increases with distance from the station and an aircraft is farther, on the average, from the VOR stations for area navigation than for radial navigation. Hence, it is desirable to improve navigational accuracy. The availability of a computer to do the triangulation computations required for area navigation and and availability of air data (airspeed and heading) suggests the possibility of using the computer to implement a Kalman filter to combine VOR/DME information and air data to obtain the desired improvement in navigational accuracy.

Problem Statement

Our objective here is to improve the accuracy of air navigation by combining VOR/DME information (from one or two stations) and air data by means of a maximum likelihood filter.

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The use of air data with the information from one VOR/DME station has been discussed by Hemesath.[1,2] However, we believe we have made several significant extensions of Hemesath's work. First of all, we feel that we have a more realistic error model for the VOR/DME measurements. Hemesath assumed additive exponentially-correlated noise in these measurements, but with such a short correlation time that it was effectively white noise. However, both the VOR and DME measurements seem to contain substantial bias errors (i.e., nearly constant errors which are, therefore, strongly correlated from one time to the next), as well as some white noise (i.e., errors that are uncorrelated from one time to the next). Modeling these bias errors is important, especially when using the VOR/DME system for area navigation. Secondly, we feel that we have implemented the maximum likelihood filter in a more straightforward manner. Thirdly, we have considered both radial and area flights while Hemesath considered only radial flights. Finally, we have studied the effects of overflying and switching VOR/DME stations during a flight.

Since more than one VOR/DME station is nearly always in sight at jet altitudes, we have also investigated the simultaneous use of two VOR/DME stations and air data. DeGroot and Larsen[3] have considered the use of two VOR's and two DME's (without air data). We have considered the use of 0, 1, or 2 VOR's; 0, 1, or 2 DME's; and air data as well as the use of 1 or 2 VOR's and 1 or 2 DME's without air data.

Error Models

Air Data

A simple kinematic model of the aircraft is‡

$$\dot{\mathbf{r}} = \mathbf{V} \tag{1}$$

where $\mathbf{r}$ is the radius vector from a reference VOR/DME station to the aircraft and $\mathbf{V}$ is the velocity of the aircraft relative to the ground. Now,

$$\mathbf{V} = \mathbf{V}_a + \mathbf{V}_w \tag{2}$$

where $\mathbf{V}_a$ is the velocity of the aircraft as determined by the onboard air data and $\mathbf{V}_w$ is the air data error. $\mathbf{V}_w$ is caused by winds and by airspeed and heading instrument errors. The winds usually dominate the airspeed and heading errors, so we did not consider it worthwhile to go into detail on sources of error for the airspeed and heading instruments.

‡ A dot denotes differentiation with respect to time and boldface denotes a vector quantity.

Letting x and y, V_{ax} and V_{ay}, and V_{wx} and V_{wy} denote the easterly and northerly components of $\mathbf{r}$, $\mathbf{V}_a$, and $\mathbf{V}_w$, respectively, from Eqs. (1) and (2), we have

$$\dot{x} = V_{ax} + V_{wx} \quad (3)$$

$$\dot{y} = V_{ay} + V_{wy} \quad (4)$$

Since we are using only a kinematic model, V_{ax} and V_{ay} will be treated as forcing functions, not as measurements. We assume that the aircraft is flying at a known, constant altitude.

We model the velocity error components V_{wx} and V_{wy} as independent, exponentially-correlated random processes. Actually V_{wy} and V_{wx} are more likely to be correlated; assuming them to be independent is conservative since they could be estimated more accurately if they were correlated. Shaping filters that generate the processes V_{wx} and V_{wy} are given by [see, e.g., Ref. 4, Sec. 11.4]

$$\dot{V}_{wx} = -(1/T_{wx})V_{wx} + (1/T_{wx})n_{wx} \quad (5)$$

$$\dot{V}_{wy} = -(1/T_{wy})V_{wy} + (1/T_{wy})n_{wy} \quad (6)$$

where n_{wx} and n_{wy} are independent white noise processes with§

$$E[n_{wx}] = 0, \qquad E[n_{wx}(t+\tau)n_{wx}(t)] = 2T_{wx}\sigma_{wx}^2\delta(\tau) \quad (7)$$

$$E[n_{wy}] = 0, \qquad E[n_{wy}(t+\tau)n_{wy}(t)] = 2T_{wy}\sigma_{wy}^2\delta(\tau) \quad (8)$$

Here σ_{wx} and T_{wx}, and σ_{wy} and T_{wy} are the rms value and the correlation time of the processes V_{wx} and V_{wy}, respectively. Only the slowly-varying, high-magnitude winds are of concern since high-frequency gusts produce no net displacement of the aircraft. Experimental data regarding wind conditions are difficult to interpret; however, a reasonable value for σ_{wx} and σ_{wy} seems to be about 40 knots while the mean values of V_{wx} and V_{wy} are taken as zero. If wind measurements are available for a particular flight path, they could be entered into the filter as nonzero mean values of V_{wx} and V_{wy}. A reasonable value for the correlation distances of both V_{wx} and V_{wy} is thought to be about 50 naut miles (nm). Thus, for an airplane travelling at a speed of 500 knots, V_{wx} and V_{wy} would have correlation times of 360 seconds.

The VOR System

A conventional VOR station¶ emits VHF signals (at 108–118 MHz) such that a reference signal and a variable signal differ by a phase angle equal to the magnetic bearing of the observation point relative to the VOR station.[6,7]

Two major sources of error in the VOR system are the misalignment of station radials and imperfect calibration of the VOR receiver. The mean values of both of these errors are approximately zero while the rms values are about 0.8 degree and 0.6 degree for the former and the latter, respectively. During the time that a given aircraft is using a particular station, these errors are essentially constant. Hence, they can be modelled as random, but constant, biases. Denoting the sum of these two errors by b_V, we have

$$\dot{b}_V = 0 \quad (9)$$

Furthermore, since the two errors are independent, b_V has a mean value of zero and an rms value, σ_{bV}, of 1.0°.

Another major source of error in the VOR system is the reflection of electromagnetic waves by fixed obstacles (trees, buildings, etc.). Also, any vertically polarized component of the VOR signal produces bearing indications which are at quadrature with the true bearing signal (horizontally polarized), thus resulting in error. This so-called polarization error can cause bearing information at a given point to vary with the heading and attitude of the aircraft. Other sources of error in the VOR system include fluctuations in the 60 Hz power supply of the VOR station, reflections from other aircraft, and receiver sensitivity to frequency and strength variations of the 30 Hz signal. For more information concerning errors in the VOR system.[6–10]

Let us denote the aggregate error in the VOR system due to all sources except radial misalignment and receiver calibration by e_V. The correlation distance for these errors appears to be about one nautical mile, which corresponds to a correlation time of 7.2 sec for an aircraft travelling at a speed of 500 knots. Since this correlation time is very short compared to the correlation time of the wind velocity components (360 sec), we model e_V as white noise

$$E[e_V] = 0, \qquad E[e_V(t+\tau)e_V(t)] = 2\sigma_{e_v}^2 T_{e_v}\delta(\tau) \quad (10)$$

where σ_{e_v} and T_{e_v} are the rms value and the correlation time of e_v, respectively. A reasonable value for σ_{e_v} is 1.0°.

The DME System

The DME airborne interrogator emits a UHF signal (960–1215 MHz) consisting of a pair of pulses. Upon receiving the signal, the ground transponder emits a pair of pulses which are received by the aircraft. Knowing the speed at which the signal travels and the elapsed time between the transmission of the interrogation signal and the reception of the reply signal, the slant range of the aircraft relative to the ground station is readily determined. In normal operation, the DME gives the slant range at a rate of 15 samples per sec.[11,12]

Errors in the DME system arise from errors in the determination of the total travelling time of the interrogation and reply signals. One such error arises from error in the time delay between the reception of the interrogation signal by the ground transponder and the emission of a reply signal. An error of similar type results from a calibration in the airborne equipment. Both of these errors are essentially constant during the time a given aircraft uses a particular station. Furthermore, these two errors are independent, each having a mean value of zero and an rms value of about 0.1 naut miles Hence, their sum, b_D, can be modelled as a random bias, that is,

$$\dot{b}_D = 0 \quad (11)$$

where b_D has a mean value of zero and an rms value, σ_{bD}, of 0.14 naut miles.

Pulse-distorting echoes cause errors in the determination of the arrival time of the interrogation pulse at the ground transponder and of the reply pulse at the airborne receiver. Also, the randomness of the amplitude of successively received interrogation signals, due to the fact that both near and far aircraft are using the same transponder, introduces error into the DME system since the time-of-arrival of a pulse is based on the instant the leading edge of the pulse reaches a fixed voltage level. Other sources of error in the DME system include transponder replies to other aircraft, pulses randomly emitted by the transponder, and receiver generated noise. For additional information concerning DME errors, see Refs. 9–12.

Let the error in the DME system due to all sources except the two sources of b_D be denoted by e_D. The correlation time of e_D is approximately 3.6 secs. Since this correlation time is very short compared to the 360 sec. correlation time of V_{wx} and V_{wy} (for an aircraft speed of 500 knots), e_D can be modelled as white noise, i.e.,

$$E[e_D] = 0, \qquad E[e_D(t+\tau)e_D(t)] = 2\sigma_{eD}^2 T_{eD}\delta(\tau) \quad (12)$$

§ $E[\cdot]$ is the expected value function and $\delta(\tau)$ is the Dirac delta function.

¶ There are several other types of VOR.[5,6] However, the only other type in general use in the U.S. is Doppler VOR which has been installed at sites where the conventional VOR yields excessive error.

Table 1 Summary of air data, VOR, and DME error models

system	error	mean value	rms value	correlation time	model
air data	V_{wx}	given	40 knots	360 sec	exponentially correlated process
	V_{wy}	given	40 knots	360 sec	exponentially correlated process
VOR	b_V	0	1.0 deg	hours	random bias
	e_V	0	1.0 deg	7.2 sec	white noise
DME	b_D	0	0.14 nm	hours	random bias
	e_D	0	0.1 nm	3.6 sec	white noise

where σ_{e_D} and T_{e_D} are the rms value and the correlation time of e_D, respectively. The mean of e_D is zero and a reasonable value for σ_{e_D} is 0.1 naut miles.

Summary

The VOR, DME, and air data error models are summarized in Table 1. The correlation times shown are for an aircraft flying at a speed of 500 knots. The mean values and rms values shown in Table 1 are thought to be reasonable values for a "typical" VOR/DME station, assuming the use of quality receivers (i.e., the type used on jet airliners).

Filter for Combining VOR/DME Information and Air Data

In this section we design a filter which combines the information from two VOR's and two DME's with air data. This filter can be used with 0, 1, or 2 VOR's; 0, 1, or 2 DME's; and air data.

System Model

We denote the VOR and DME measurements from station 1 and station 2 as V_1 and D_1, and V_2 and D_2, respectively. Figure 1 shows the VOR/DME station configuration. We define

$$\mathbf{R}_{12} \triangleq \mathbf{R}_1 - \mathbf{R}_2 \tag{13}$$

and denote the easterly, northerly, and vertical components of $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_{12}$ by (x_1,y_1,h_1), (x_2,y_2,h_2), and (x_{12},y_{12},h_{12}), respectively. Using these definitions and the error models derived previously, we have the following system model:

State Equations: $\dot{x} = Fx + f + n$

That is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{V}_{wx} \\ \dot{V}_{wy} \\ \dot{b}_{V1} \\ \dot{b}_{D1} \\ \dot{b}_V{}^2 \\ \dot{b}_D{}^2 \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\left(\frac{1}{T_{wx}}\right) & 0 & (4\times 4) \\ 0 & 0 & 0 & -\left(\frac{1}{T_{wy}}\right) & \\ \hline & & 0 & & \\ & & (4\times 8) & & \end{array}\right] \begin{bmatrix} x_1 \\ y_1 \\ V_{wx} \\ V_{wy} \\ b_{V1} \\ b_{D1} \\ b_{V2} \\ b_{D2} \end{bmatrix} + \begin{bmatrix} V_{ax} \\ V_{ay} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{n_{wx}}{T_{wx}} \\ \frac{n_{wy}}{T_{wy}} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

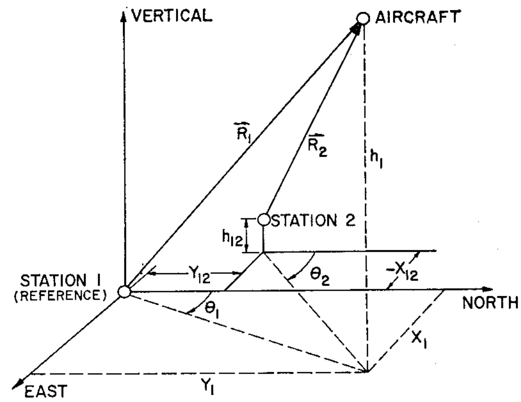

Fig. 1 VOR/DME station configuration.

where the subscripts 1 and 2 indicate the station with which a quantity is associated.

Measurement Equations: $z = h(x) + v$

That is,

$$\begin{bmatrix} V_1 \\ D_1 \\ V_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \arctan(x_1/y_1) + b_{V1} \\ [x_1{}^2 + y_1{}^2 + h_1{}^2]^{1/2} + b_{D1} \\ \arctan[(x_1 - x_{12})/(y_1 - y_{12})] + b_{V2} \\ [(x_1 - x_{12})^2 + (y_1 - y_{12})^2 + (h_1 - h_{12})^2]^{1/2} + b_{D2} \end{bmatrix} + \begin{bmatrix} e_{V1} \\ e_{D1} \\ e_{V2} \\ e_{D2} \end{bmatrix} \tag{15}$$

Linearization about the Estimated Path

The measurement Eq. (15) must be linearized in order to use the Kalman-Bucy filtering theory.[13,14] In operation, Eq. (15) would be linearized about the current estimate of the path. For purposes of simulation, we linearize here about a predetermined nominal path. Denoting the nominal values of x and z by $\bar{x}$ and $\bar{z}$, respectively, we have

$$\begin{aligned} x &= \bar{x} + \tilde{x} \\ z &= \bar{z} + \tilde{z} \end{aligned} \tag{16}$$

where $\tilde{x}$ and $\tilde{z}$ represent small perturbations of x and z about their nominal values. Thus, using Eqs. (14, 15, and 16), the following set of linearized perturbation equations results:

$$\begin{aligned} \dot{\tilde{x}} &= F\tilde{x} + n \\ \tilde{z} &= H\tilde{x} + v \end{aligned} \tag{17}$$

where H is the derivative of $h(x,t)$ with respect to x, evaluated at $\bar{x}$, i.e.,

$$H = \left.\frac{\partial h}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} \frac{\bar{y}_1}{A}, & \frac{-\bar{x}_1}{A}, & 0,0,1,0,0,0 \\ \frac{\bar{x}_1}{[A + h_1{}^2]^{1/2}}, & \frac{\bar{y}_1}{[A + h_1{}^2]^{1/2}}, & 0,0,0,1,0,0 \\ \frac{\bar{y}_1 - y_{12}}{B}, & \frac{x_{12} - \bar{x}_1}{B}, & 0,0,0,0,1,0 \\ \frac{\bar{x}_1 - x_{12}}{[B + (h_1 - h_{12})^2]^{1/2}}, & \frac{\bar{y}_1 - y_{12}}{[B + (h_1 - h_{12})^2]^{1/2}}, & 0,0,0,0,0,1 \end{bmatrix} \tag{18}$$

and

$$A = \bar{x}_1{}^2 + \bar{y}_1{}^2, \qquad B = (\bar{x}_1 - x_{12})^2 + (\bar{y}_1 - y_{12})^2 \tag{19}$$

Let t_o denote the time that the filter is initialized. Then, using the previously described error models, we have

$$E[n(t)] = E[v(t)] = E[n(t)v^T(\tau)] = 0 \tag{20}$$

$$E[\tilde{x}(t_o)] = E[\tilde{x}(t_o)n^T(t)] = E[\tilde{x}(t_o)v^T(t)] = 0 \tag{21}$$

$$\begin{aligned} E[n(t+\tau)n^T(t)] &= Q\delta(\tau) \\ &= \mathrm{diag}\{0, 0, 2\sigma_{wx}{}^2/T_{wx}, 2\sigma_{wy}{}^2/T_{wy}, 0, 0, 0, 0\}\delta(\tau) \end{aligned} \tag{22}$$

$$\begin{aligned} E[v(t+\tau)v^T(t)] &= R\delta(\tau) \\ &= \mathrm{diag}\{2T_{eV1}\sigma_{eV1}{}^2, 2T_{eD1}\sigma_{eD1}{}^2, \\ &\qquad 2T_{eV2}\sigma_{eV2}{}^2, 2T_{eD2}\sigma_{eD2}{}^2\}\delta(\tau). \end{aligned} \tag{23}$$

The error covariance matrix, P, at t_o, is given by

$$P(t_o) = E[\tilde{x}(t_o)\tilde{x}^T(t_o)] \tag{24}$$

where all the elements of $P(t_o)$ are zero except for those on the main diagonal, $P_{12}(t_o)$. (the element in the first row and second column of $P(t_o)$), and $P_{21}(t_o)$. In particular,

$$\begin{aligned} \mathrm{diag}\, P(t_o) = \{&\sigma_{x1}{}^2(t_o), \sigma_{y1}{}^2(t_o), \sigma_{wx}{}^2, \sigma_{wy}{}^2, \sigma_{bV1}{}^2, \\ &\sigma_{bD1}{}^2, \sigma_{bV2}{}^2, \sigma_{bD2}{}^2\} \end{aligned} \tag{25}$$

$$\sigma_{x1y1}{}^2 = P_{12}(t_o) = P_{21}(t_o) = E[\tilde{x}_1(t_o)\tilde{y}_1(t_o)]. \tag{26}$$

Note that the values of all quantities appearing in Eqs. (22–26), except $\sigma_{x1}(t_o)$, $\sigma_{y1}(t_o)$, and $\sigma_{x1y1}(t_o)$, can be readily determined from the previous discussions.

Initial Position Errors

From Fig. 1, it is easily seen that

$$\begin{aligned} x_1 &= [R_1{}^2 - h_1{}^2]^{1/2}\sin(\theta_1) \\ y_1 &= [R_1{}^2 - h_1{}^2]^{1/2}\cos(\theta_1) \end{aligned} \tag{27}$$

Denoting the total errors in the VOR and DME measurements from station 1 by E_{V1} and E_{D1}, respectively, the values of x_1 and y_1 determined from the VOR/DME measurements are

$$\begin{aligned} x_{1m} &= [(R_1 + E_{D1})^2 - h_1{}^2]^{1/2}\sin(\theta_1 + E_{V1}) \\ y_{1m} &= [(R_1 + E_{D1})^2 - h_1{}^2]^{1/2}\cos(\theta_1 + E_{V1}) \end{aligned} \tag{28}$$

Subtraction of Eqs. (27) from Eqs. (28) yields:

$$\begin{aligned} E_{x1} = x_{1m} - x_1 &= [(R_1 + E_{D1})^2 - h_1{}^2]^{1/2}\sin(\theta_1 + E_{V1}) - \\ &\qquad [R_1{}^2 - h_1{}^2]^{1/2}\sin(\theta_1) \\ E_{y1} = y_{1m} - y_1 &= [(R_1 + E_{D1})^2 - h_1{}^2]^{1/2}\cos(\theta_1 + E_{V1}) - \\ &\qquad [R_1{}^2 - h_1{}^2]^{1/2}\cos(\theta_1) \end{aligned} \tag{29}$$

For $|E_{D1}| \ll R_1$ and $|E_{V1}| \ll 1$, these relations may be approximated by

$$\begin{aligned} E_{x1} &= (1 - h_1{}^2/2R_1{}^2)[R_1\cos(\theta_1)E_{V1} + \sin(\theta_1)E_{D1}] \\ E_{y1} &= (1 - h_1{}^2/2R_1{}^2)[\cos(\theta_1)E_{D1} - R_1\sin(\theta_1)E_{V1}] \end{aligned} \tag{30}$$

From Eqs. (30) and the fact that E_{V1} and E_{D1} are uncorrelated, it follows that

$$\begin{aligned} \sigma_{x1}{}^2 = E[E_{x1}{}^2] &= (1 - h_1{}^2/2R_1{}^2)^2[R_1{}^2\cos^2(\theta_1)\sigma_{V1}{}^2 + \\ &\qquad \sin^2(\theta_1)\sigma_{D1}{}^2] \\ \sigma_{y1}{}^2 = E[E_{y1}{}^2] &= (1 - h_1{}^2/2R_1{}^2)^2[R_1{}^2\sin^2(\theta_1)\sigma_{V1}{}^2 + \\ &\qquad \cos^2(\theta_1)\sigma_{D1}{}^2] \\ \sigma_{x1y1}{}^2 = E[E_{x1}E_{y1}] &= (1 - h_1{}^2/2R_1{}^2)^2\sin(2\theta_1) \times \\ &\qquad (\sigma_{D1}{}^2 - R_1{}^2\sigma_{V1}{}^2)/2 \end{aligned} \tag{31}$$

where

$$\begin{aligned} \sigma_{V1}{}^2 &= E[E_{V1}{}^2] = \sigma_{bV1}{}^2 + \sigma_{eV1}{}^2, \\ \sigma_{D1}{}^2 &= E[E_{D1}{}^2] = \sigma_{bD1}{}^2 + \sigma_{eD1}{}^2 \end{aligned} \tag{32}$$

The initial position variances and covariance, $\sigma_{x1}{}^2(t_o)$, $\sigma_{y1}{}^2(t_o)$ and $\sigma_{x1y1}{}^2(t_o)$, are calculated from Eqs. (31) and (32) by substituting the values of θ_1 and R_1 at t_o.

Filter Equations

The Kalman-Bucy filter equations for the continuous-time, discrete-data system described by Eqs. (13–26) are: Between measurements,

$$\dot{\hat{x}} = F\hat{x} + f \tag{33}$$

$$\dot{P} = FP + PF^T + Q \tag{34}$$

At a measurement,

$$\hat{x}_+ = \hat{x}_- + K[z - \hat{h}(\hat{x}_-)] \tag{35}$$

$$P_+ = (I - KH)P_-(I - KH)^T + KRK^T \tag{36}$$

$$K = P_-H^T(HP_-H^T + R)^{-1} \tag{37}$$

where P is the error covariance matrix, I is the 8×8 identity matrix, and the $-$ and $+$ mean before and after the measurement, respectively.

A solution to Eq. (34) is of the form [see, e.g., Ref. 4, p. 453]

$$P(t) = \phi(t, t_1)P(t_1)\phi^T(t, t_1) + \int_{t_1}^{t}\phi(t, \tau)Q(\tau)\phi^T(t, \tau)d\tau \tag{38}$$

where $\phi(t, t_1)$ is the transition matrix associated with the system $\dot{x} = Fx$ and satisfies the equation

$$d\phi(t, t_1)/dt = F(t)\phi(t, t_1), \qquad \phi(t_1, t_1) = I \tag{39}$$

For the system under consideration, the following expression for the transition matrix can be found by solving Eq. (39)

$$\phi(t, t_1) = \left[\begin{array}{cccc|c} 1 & 0 & T_{wx}(1 - M) & 0 & \\ 0 & 1 & 0 & T_{wy}(1 - N) & 0 \\ 0 & 0 & M & 0 & (4 \times 4) \\ 0 & 0 & 0 & N & \\ \hline & & 0 & & I \\ & & (4 \times 4) & & (4 \times 4) \end{array}\right] \tag{40}$$

where

$$M = \exp\{-(t - t_1)/T_{wx}\}, \quad N = \exp\{-(t - t_1)/T_{wy}\}, \tag{41}$$
$$I = \text{identity matrix}$$

Define

$$J \triangleq \int_{t_1}^{t}\phi(t, \tau)Q(\tau)\phi^T(t, \tau)d\tau \tag{42}$$

Substituting from Eqs. (22, 40, and 41) into Eq. (42) and integrating yields:

$$\begin{aligned} &J(i, j) = 0 \quad \text{for all} \quad i, j = 1, 2, \ldots, 8 \\ &\text{except} \\ &J(1,1) = T_{wx}\sigma_{wx}{}^2[2(t - t_1) - T_{wx}(3 - M)(1 - M)] \\ &J(1,3) = J(3,1) = T_{wx}\sigma_{wx}{}^2(1 - M)^2 \\ &J(2,2) = T_{wx}\sigma_{wy}{}^2[2(t - t_1) - T_{wy}(3 - N)(1 - N)] \\ &J(2,4) = J(4,2) = T_{wy}\sigma_{wy}{}^2(1 - N)^2 \\ &J(3,3) = \sigma_{wx}{}^2(1 - M^2) \\ &J(4,4) = \sigma_{wy}{}^2(1 - N^2) \end{aligned} \tag{43}$$

Thus, from Eqs. (40–43), Eq. (38) is seen to constitute a closed-form solution to Eq. (34) which involves only the multiplication and addition of matrices.

Summary

The design of a filter to combine VOR/DME information and air data was based on a kinematic model of the motion of the aircraft projected onto the local horizontal plane. It estimates two components of position relative to station 1, the two horizontal components of wind velocity, and the VOR and DME biases associated with each station. The initial position error statistics are those obtained by taking a VOR/DME reading from the station relative to which we are estimating position. When the aircraft stops using a VOR/DME station and begins to use another, the bias error estimates must be re-initialized.

Estimator for Combining Information From Two VOR/DME's Without Air Data

Combining VOR/DME information without air data reduces to the problem of finding the maximum likelihood estimates of x_1 and y_1 at each measurement time. The estimator equations for the case of combining the information from two VOR's and two DME's are [see, e.g., Ref. (4), Sec. 12.2]

$$\hat{x}_e = \bar{x}_e + P_e H_e^T R_e^{-1}[z - h(\bar{x})] \tag{44}$$

$$P_e = [H_e^T R_e^{-1} H_e]^{-1} \tag{45}$$

with

$$x_e = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \qquad P_e = E[(\hat{x}_e - x_e)(\hat{x}_e - x_e)^T] \tag{46}$$

$$R_e = \mathrm{diag}\{\sigma_{V1}^2, \sigma_{D1}^2, \sigma_{V2}^2, \sigma_{D2}^2\} \tag{47}$$

and $H_e = 4 \times 2$ matrix consisting of the first two columns of H where σ_{V1}^2 and σ_{D1}^2 are defined in Eqs. (32) and

$$\sigma_{V2}^2 = \sigma_{bV2}^2 + \sigma_{eV2}^2 \tag{48}$$

$$\sigma_{D2}^2 = \sigma_{bD2}^2 + \sigma_{eD2}^2. \tag{49}$$

The estimator for investigating any combination of 0, 1, or 2 VOR's and 0, 1, or 2 DME's is a special case of this estimator which is obtained by deleting the measurements not taken.

Simulation Results

A computer program was written to calculate the error covariance matrix for any nominal flight path consisting of a series of straight line segments. The inputs to the program are: a) the model parameters, b) the latitude, longitude, and altitude of each of the VOR/DME stations to be used, and c) the latitude and longitude of each of the switching points, i.e., points along the flight path where the aircraft tunes in a new VOR/DME station or changes its directional heading. The outputs of the program are filter gains and the RMS errors in the estimates of position, velocity, and VOR/DME biases.

Using this program, the rms error histories for various flight paths were calculated. The results presented here are for an aircraft flying due east at an altitude of 33,000 ft with a speed of 500 knots and sampling the VOR/DME data every 10 secs. The filter performance does not vary substantially for sampling times of from one second to one minute.

Rms Errors

The rms position errors which occur when using the information from one VOR/DME with and without air data are shown in Figs. 2 and 3 for a radial and an area flight, respectively. The improvement which results when air data are added is larger for area than for radial flights. This is due to the fact that the filter can estimate the VOR bias more accurately for area than for radial flights, while just the opposite is true for the DME bias. This is shown in Figs. 4 and 5, respectively. Since the position error due to the VOR bias is larger, the factor of improvement is greater for area flights.

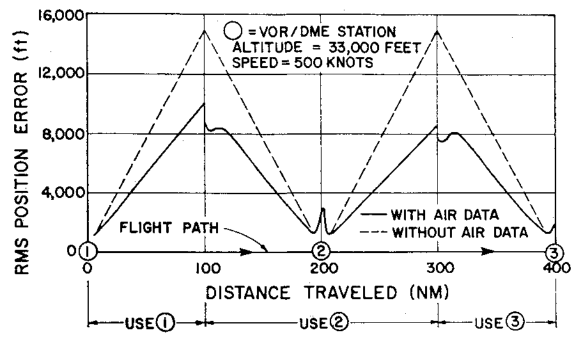

Fig. 2 Rms position errors for a radial flight using one VOR/DME with and without air data.

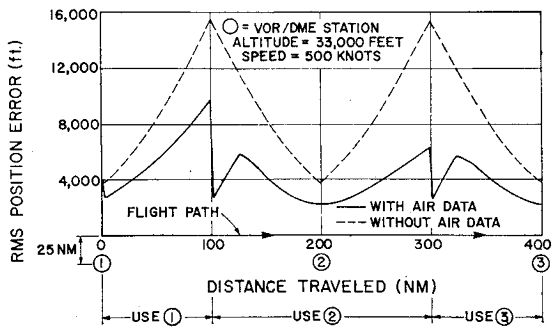

Fig. 3 Rms position errors for an area flight using one VOR/DME with and without air data.

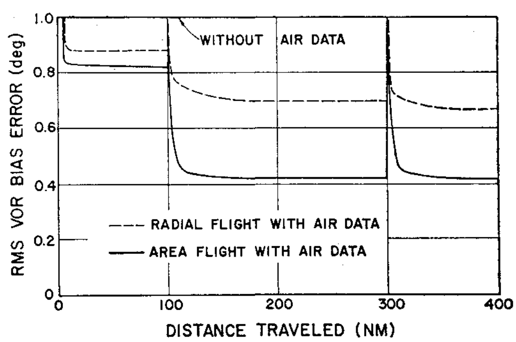

Fig. 4 Comparison of rms VOR bias errors for the radial and area flights of Figs. 2 and 3.

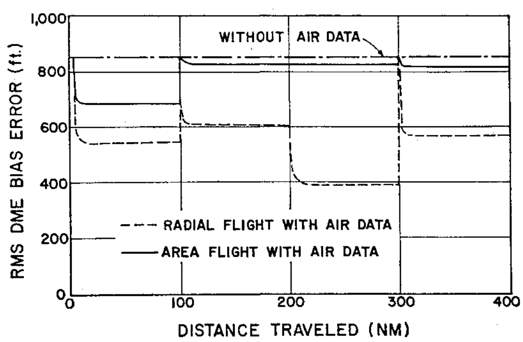

Fig. 5 Comparison of rms DME bias errors for the radial and area flights of Figs. 2 and 3.

The sharp decreases in rms position errors in Figs. 2 and 3 at the points where the aircraft switches from one VOR/DME station to another are of interest. The decreases which occur in the radial flight (as well as a portion of the decreases for the area flight) are due to a transient effect which is introduced when, upon tuning in a new station, the variances of the biases are re-initialized and the off-diagonal terms involving the biases are set to zero in the covariance matrix. The larger decreases occurring in the area flight are explained by the fact that the DME position information is more accurate than the VOR position information (except when very near the station), and the fact that the lines-of-sight to the new and old stations at the switching points for the area flight (Fig. 3) are not parallel.

Also of interest is the rapidity with which the rms VOR and DME bias errors tend to nearly constant values after switching to a new station (see Figs. 4 and 5). The reason for this is that the aircraft has a rather good estimate of its position as a result of filtering data from the previous station; and hence, as the aircraft switches to the new station, this enables the filter to estimate the new bias errors quickly.

At elevation angles above 60°, the VOR signals are unusable due to excessive interference. Hence, when using only VOR/DME information, the rms position error gets large when overflying a VOR/DME station. However, when air data are added, the rms position error remains relatively small as shown in Fig. 2.

In Fig. 6, the rms position errors for various combinations of the information from two VOR/DME's and air data are shown. When a second VOR is added to the case of one VOR and two DME's or the case of one VOR, two DME's and air data, the decreases in the rms position error are negligible (less than five feet).

Since the position error due to VOR error is much greater than that due to DME error, the best position accuracy, when using the two stations, occurs when the crossing angle between the lines-of-sight from the aircraft to the two stations is 90°. For the flight path of Fig. 6, the crossing angle varied from about 60° to 120°. As can be seen from Fig. 6, the rms position errors increase as the crossing angle deviates from 90°. The present network of VOR/DME stations would allow an aircraft flying at jet altitudes (24,000 to 45,000 ft) to choose stations so that the crossing angle lies between 60 and 120 degrees almost everywhere in the U.S.[3]

Factors of improvement in rms position error over the use of a single VOR/DME are shown in Table 2 for various combinations of information. These factors were calculated for a point halfway between the second and third stations because at this point the error for the case of using a single VOR/DME is maximum; and hence, the factor of improvement at this point is of prime importance. The case of using two DME's without air data is not included in Table 2 because

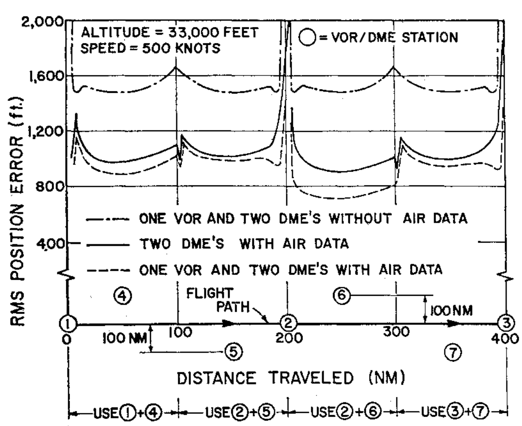

Fig. 6 Rms position errors for various combinations of the information from two VOR/DME's and air data.

Table 2 Approximate factors of improvement in rms position accuracy over the use of a single VOR/DME for various combinations of VOR/DME information and air data

combination of information	factor of improvement
1 VOR, 1 DME, air data (radial flight)	1.8
1 VOR, 1 DME, air data (area flight)	2.5
1 (or 2) VOR's, 2 DME's	9
2 DME's, air data	15
1 (or 2) VOR's, 2 DME's air data	19

there are generally two position fixes possible in this case. However, the addition of a VOR measurement to two DME measurements resolves the ambiguity, although it does not substantially improve the accuracy of the position fixes. From these results, we see that the navigational accuracy resulting from the use of a given combination of VOR/DME information is improved by roughly a factor of two by the addition of air data.

Rms velocity errors were reduced by a factor of roughly two when one VOR, one DME, and air data are combined, and by a factor of about three for combinations involving the information from two VOR/DME's and air data.

Jet flights along approved radial and area routes between San Francisco and Chicago were simulated and the results checked those presented here between the second and third stations of these short flights.

Sensitivity Analysis

The design of the filter was based on "nominal" values for the error model parameters. Since some of these parameters are quite uncertain, it is desirable to establish how the filter performs when the error statistics are not nominal.

There is considerable uncertainty in the correlation times as well as in the rms values of the air data errors (σ_{wx} and σ_{wy}) and the white noise component of the VOR error (σ_{ev}). In order to determine the sensitivity of the filter performance to variations in these error parameters, certain parameters were assumed to be "non-nominal," and the performance degradation, i.e., the difference between the factors of improvement in rms position accuracy obtained from the nominal filter and the optimal filter (the filter designed using the fact that certain parameters were not nominal) over the nominal factor of improvement, was calculated. The results are shown in Table 3. The flight path was that of Fig. 2 with the performance degradation calculated at the midpoint between the second and third stations. From these results, it appears that the performance of the (nominal) filter may be insensitive to fairly wide variations in error model parameters.

Table 3 Filter sensitivity to variations in error parameters

Non-nominal parameters	Performance degradation (%)
All correlation times halved	2
All correlation times doubled	3
$\sigma_{wx} = 20$ knots	4
$\sigma_{wy} = 60$ knots	
$\sigma_{wx} = 60$ knots	11
$\sigma_{wy} = 20$ knots	
$\sigma e_V = 0.5$ deg.	5
$\sigma e_V = 2.0$ deg.	8

Conclusions

Combining VOR/DME information (from one or two stations) and air data (airspeed and heading) by means of a maximum likelihood filter results in substantial improvements in navigational accuracy over than obtained by the use of a single VOR/DME, as is the current practice.

The addition of air data to the information from a single VOR/DME station reduces the rms position and velocity errors by a factor of about 2. Much of the improvement due to the use of air data results from the ability of the filter to estimate the bias errors associated with each VOR/DME station that the aircraft uses in the course of its flight. Estimates of VOR bias error are better for area than for radial flight paths while the opposite is true for the DME bias error.

The use of information from two VOR/DME stations with air data was found to yield large factors of improvement (from 15 to 20) in the rms position error over the use of a single VOR/DME station, while the rms velocity error was reduced by a factor of roughly 3. As far as position accuracy is concerned, at most one VOR station need be used. Since the sampling time required of the VOR/DME signals is on the order of 10 to 15 sec and a DME receiver can be tuned in a few seconds, it appears possible that an automatic time-sharing device could be designed for the present DME receivers to use information from two stations.

The velocity estimates, made when using air data, could be used for flight guidance. Also, for portions of the flight where VOR/DME signals are lost (for example, when passing directly over a station), a dead-reckoning mode continues using the air data.

The filter performance was found to be rather insensitive to wide variations in error model parameters. However, more information, particularly regarding air data errors, should be acquired.

The filter gains associated with the VOR and DME measurements were found to be highly dependent upon the location of the VOR/DME stations relative to the flight path. Because of the nature of these gains, the search for a suboptimal filter with constant and linearly-varying gains proved fruitless.

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